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Improved Pareto Set Based Method for Solving τ-Objective Linear Programming Problems

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Abstract

A basic idea in mathematics, convex combinations are especially important in linear algebra, convex analysis, and optimization. They offer a method for creating new points from a given set while maintaining convexity, which is a crucial characteristic in many applied and mathematical domains. In this paper, we improved and generalized an idea for solving multi-objective function to find a compromise solution. The method focused on using convex combination of some points namely, positive efficient points which is a new definition. The results providing acceptable solutions for decision-makers. Comparing it with other methods, the method gives range of solutions as well as, can be used in the case of the individual optimal solutions are on distinct extreme points.

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1. Introduction

Optimization, especially linear programming is extensively utilized across various sectors to efficiently allocate limited resources in order to meet customer demands [1]. They appear in many applications in real life situations. Mathematical programming used for travelling salesman to improve results for different problems [2]. Also, fractional linear programming has a great role in applications in fuzzy environment [3, 4]. A linear programming method is utilized for scheduling jobs on machines to determine the makespan while considering the precedence of operations in a flow shop setting [5]. Lastly, A significant application of linear programming lies in statistics for the estimation of regression parameters [6]. Multi-objective linear programming (MOLP) is a crucial area of optimization that addresses decision-making challenges involving multiple conflicting objectives. Its primary aim is to identify solutions that strike a balance among these objectives, thereby providing decision-makers with a range of Pareto-optimal options. Since its inception, MOLP has been extensively utilized across various sectors, including economics, engineering, environmental planning, logistics, and healthcare. Sen [7] discovered a novel method for addressing multi-objective linear programming problems and proposes a strategy for formulating the multi-objective function, given the constraint that the optimal value of each individual problem greater than zero. Subsequently, numerous researchers endeavored to address multi-objective linear programming problems. Sulaiman and Sadiq [8] investigated the multi-objective function by addressing the multi-objective programming problem through the application of mean and median value. Zahidul and Asadujjaman [9] proposed a novel average method for addressing multi-objective linear programming problems by employing various mean techniques. Sen employed a statistical methodology to address multi-objective programming (MOPP) challenges. Previous studies have not identified an optimal solution; instead, they provide a satisfactory solution to the problem. Recent studies have concentrated on addressing multi-objective challenges across various environments. Tougma, and Somé [10] addressing multi-objective optimization challenges with inequality constraints through the utilization of an augmented Lagrangian function. Wajahat et al. [11] solved a combined problem by several approaches. Getaye and Adane [12] compare mean and median approaches to solve multi-objective linear fractional programming problem. This study introduces the enhanced convex combination technique to achieve approximate solution for a multi-objective linear programming problem. This technique provides a decision maker with an excellent opportunity to select their preferred results.

2. Model of τ-Objective Linear Programming Problems (τ-OLPP)

The model can be formulated as optimize S-objectives simultaneously under linear constraints as follows

$$\left.\begin{array}{ll} Max.\ z_i = Co_{it}X + \gamma_i & i=1,\ldots,r \\ Min.\ z_i = Co_{it}X + \gamma_i & i=r+1,\ldots,s \\ \text{Subject to:} & \\ AX\ [\geq = \leq]\ B & \\ X \geq 0, & \end{array}\right\} \qquad (1)$$

where X is $X_{(n\times 1)}$vector, Co is $Co_{(1\times n)}$ vector of constants, B is $b_{(m\times 1)}$ vector of constants, r is the number of objective functions to be maximized, s the number of objective functions to maximized plus minimized, $(s-r)$ is the number of objectives that is to be minimized, $\tau = (r+s)$, A is A $(m\times n)$ vector matrix, $\gamma_i(i=1,\ldots,s)$ are constants, $Co_{it}X + \gamma_i$ $(i=1,\ldots, s)$ are linear factors for all feasible solutions. Therefore, the problem is called to be τ-objective linear programming problem (τ-OLPP) in case of all the functions are linear [13].

3. Methods for Solving τ-OLPP

There are different methods for solving τ-OLPP, which combining individual objective functions in to determine a compromise solution. Two directions are as follows: After finding the optimal solution individually for each objective, say Z_i^*, $i=1,\ldots,\tau$, at x_i^*, $i=1,\ldots,\tau$ respectively.

a) Combine Z_i^* by some methods to find compromise solution, see [7, 14]

b)Combine x_i^* by some methods to find compromise solution [15].

It is important to mention that most of the methods are focused in a.

4. New Directions

Zimmerman presented a famous paper [16] for fuzzy linear programming approaches to the linear vectormaximum problem. Showed that fuzzy linear programming gives always efficient solutions. Also, [15] used the idea of convex combination for bi-objective LPP. The shortage of the two above methods is: [16] works only for bi-objective, and [15] choose some cases out of many cases as we will show later. In this paper we improved them in the context of solving τ-OLPP, and find the best choice for convex combination.

Definition 1. Let S be a feasible region. For a τ-objective optimization problem (1), a solution $\theta*$ is said to be an efficient solution iff $\nexists\ \theta \in$ S such that $z_i(\theta) < z_i(\theta*)$, for all $i = 1, 2, \ldots, \tau$, for at least one i [17].

Definition 2. A compromise solution of the (τ-OLPP) is the feasible solution preferred by the decision maker (DM) by considering all objectives [18].

Definition 3. A convex combination in convex geometry is a linear combination of points with all non-negative coefficients and a sum of 1. In other words, a convex combination of a finite number of points $x_1, \ldots, x_n$ in a real vector space is a point of type $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_2$, where the real values αi fulfill $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_n$. For instance, the line segment between any two points includes their convex combination [19].

Definition 4. A set of all efficient solution is called Pareto set.

5. Mathematics of Convex Combination CC

Consider a τ-OLPP and the feasible region can be found easily. Let the number of extreme points (basic feasible solutions) of (1) is equal to K.

Proposition 1: Number of efficient solutions of (1) is less than "≤" K.

Proof:

Let (a, b) and (c, d) be two extreme points in feasible region of (1). If $a < c$ and $b < d$ which is possible of extreme point, because they lie on the boundary. Therefore (a, b) is not efficient point. ■

Now, let number of efficient solutions is K_1. Since the convex combination takes two points from K_1 points, so the cases are $C_2^{K_1} = \frac{K_1!}{2!\,(K_1-2)!}$. So, if K_1= 4, there are 6 possible cases for each λ. There are infinite solutions for the problem. To reduce the cases, we present new concepts and definitions.

Definition 5. (New) A feasible point $X^* = (x^*, y^*)$ is strongly efficient for (1) if there is no any feasible point $X = (x, y)$ such that $x \le x^*$; $y \le y^*$ where at least one of the inequalities is strict. If one of the inequalities is with equality then it is weak efficient.

Example 1. Consider the following set of points $S = \{(4,5), (3,1), (4,2), (7,3)\}$. The point $X^* = (x^*, y^*) = (3,1)$ is strongly efficient because $\nexists$ any x in the set ≤ 3, nor $y \le 1$. For the set $S_1 = \{(4,5), (3,1), (4,1), (7,3)\}$. The point $(3,1)$ is weak efficient because $\exists$ any y in the set $= 1$ which is point $(4,1)$.

Definition 6. (New) An efficient point is called positive if $z_i(X^*) > 0$, for all $i = 1, 2, \dots, \tau$.

Example 2. Let $S_2 = \{(4,7), (7,6), (8,3), (9,5)\}$. The efficient points are $(4,7), (7,6), (8,3)$. Since we have bi-criteria function. Consider two criteria z_1 and z_2. If

Efficient points	z_1	z_2
$(4,7)$	Positive	Positive
$(7,6)$	Negative	Positive
$(8,3)$	Positive	Negative

Then $(4,7)$ is positive efficient point.

6. Improved Approach

Step 1: Find optimal value for each $(i = 1, 2, \dots, \tau)$. Characterize the extreme points $e_1, e_2, \dots, e_k$. Find strongly efficient points.
Step 2: Determine the positive efficient solutions K_2. From K_2 choose two points, say (p, q) and (p^*, q^*)
Step 3: Find a new point P* which is a convex combination of (p, q) and (p^*, q^*), so $P^* = \lambda_1 (p, q) + \lambda_2 (p^*, q^*)$, where $\lambda_1, \lambda_2 \ge 0$, and $\lambda_1 + \lambda_2 = 1$. Construct a linear system.
Step 4: The decision-maker choose values of λ_1, λ_2 based on his circumstances.
Step 5: End.

6.1. Possible Cases

The convex combination needs 2 points. So, we have

a) If K_2 =1, choose a weak efficient point.
b) If $K_2 \ge 2$, choose two with large z_i, for all $i = 1, 2, \dots, \tau$.

7. Illustrated Examples

Example 3.

$$Max\ Z_1 = 3x_1 + x_2$$
$$Max\ Z_2 = -2x_1 + x_2$$

subject to

$$-x_1 + x_2 \le 3$$
$$x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$x_1 \le 5$$
$$x_1, x_2 \ge 0.$$

Figure 1 shows the feasible region of this problem. We have 5 points $(K = 5)$ which are basic feasible solutions as follows: (5, 1), (5, 0), $(0,3), (1,4), (2,4)$.

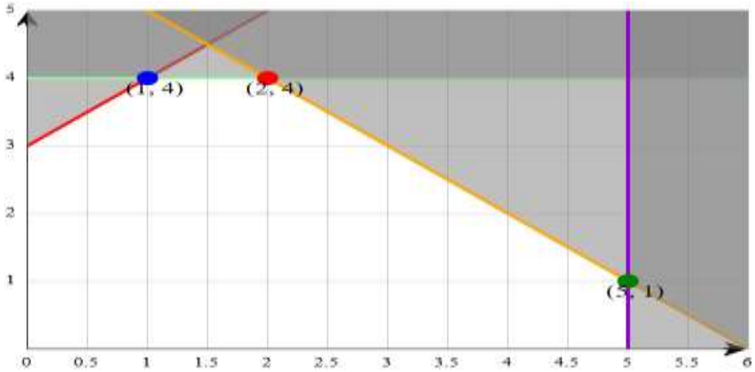

Figure 1: Graphical solution for the objective function

To apply our algorithm. According to the definition 4, and 5 the positive efficient solutions are: $Z_1\,(0,3) = 3$, , $Z_1\,(1,4) = 7$, $Z_1\,(2,4) = 10$, $Z_2\,(0,3) = 3$, , $Z_2(1,4) = 2$, $Z_2\,(2,4) = 0$. The strong efficient point is only $(2,4)$. Therefore, we choose point $(1,4)$ as weak efficient. Utilizing the convex combination definition, we obtain the subsequent linear system:

$$x_1^* = 1\lambda_1 + 2\lambda_2$$
$$x_2^* = 4\lambda_1 + 4\lambda_2$$
$$\lambda_1 + \lambda_2 = 1.$$

Table 1 shows some known cases according to λ_1, λ_2. For $\lambda_1 = 1, \lambda_2 = 0$ or vice versa, we get the points themselves which guarantee the feasibility for the new point

λ_1	λ_2	x_1^*	x_2^*	Z_1	Z_2
1	0	1	4	7	2
0	1	2	4	10	0
0.5	0.5	1.5	4	4.6	3.6
0.1	0.9	1.9	4	9.7	0.2
0.9	0.1	1.1	3.8	7.1	1.6

Table 1. Results of example 3 for known cases

Example 4. (Zimmerman example)

$$\text{Max } Z_1 = -x_1 + 2x_2$$
$$\text{Max } Z_2 = 2x_1 + x_2$$

subject to

$$-x_1 + 3x_2 \le 21$$
$$x_1 + 3x_2 \le 27$$
$$4x_1 + 3x_2 \le 45$$
$$3x_1 + x_2 \le 30$$
$$x_1, x_2 \ge 0.$$

Figure 2 shows the feasible region of this problem. We have 5 points ($K = 5$) which are basic feasible solutions as follows: $(0,7), (3,8), (6,7), (9,3)$ and $(10,0)$.

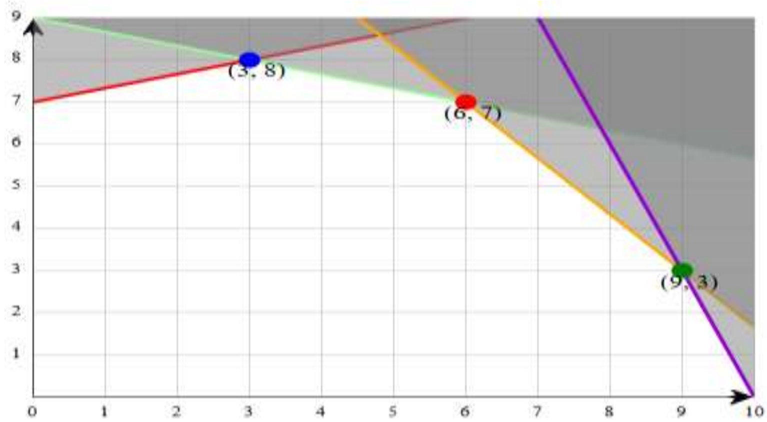

Figure 2. Graphical solution for the objective function

To apply our algorithm, at first, we find the feasible region, and then the extreme points which are: $(0,7), (3,8), (6,7), (9,3)$ and $(10,0)$, with the following objective values. $Z_1(0,7) = 14, Z_1(3,8) = 13$, $Z_1(6,7) = 8$, $Z_1(9,3) = -3$ and $Z_1(10,0) = -10$. $Z_2(0,7) = 7$, $Z_2(3,8) = 14$, $Z_2(6,7) = 19$, $Z_2(9,3) = 21$ and $Z_2(10,0) = 20$. The convex combination gives $C_2^5 = \frac{5!}{2!} =$ 10 cases. For illustration we check one case.

Case 1: Points $(0,7)$, $(3,8)$

The linear system is

$$x_1^* = 3\lambda_2$$
$$x_2^* = 7\lambda_1 + 8\lambda_2$$
$$\lambda_1 + \lambda_2 = 1$$

Table 2 shows the results for different cases

λ_1	λ_2	x_1^*	x_2^*	Z_1	Z_2
1	0	3	8	13	14
0	1	0	7	14	7
0.5	0.5	1.5	7.5	13.5	10.5
0.1	0.9	2.7	7.9	13.1	13.3
0.9	0.1	0.3	7.1	13.9	7.7

Table 2. Results of example 4 for known cases

The 9 possible cases are shown in Table 3 for some values of λ_1, λ_2

Cases	$\lambda_1 = 0.5$ $\lambda_2 = 0.5$	$\lambda_1 = 0.1$ $\lambda_2 = 0.9$	$\lambda_1 = 0.15$ $\lambda_2 = 0.85$	Zimmermann
1	$Z_1 = 13.5$ $Z_2 = 10.5$	$Z_1 = 13.1$ $Z_2 = 13.3$	$Z_1 = 13.5$ $Z_2 = 12.95$	$Z_1 = 9.61$ $Z_2 = 17.38$
2	$Z_1 = 11$ $Z_2 = 13$	$Z_1 = 8.6$ $Z_2 = 17.8$	$Z_1 = 8.9$ $Z_2 = 17.2$	
3	$Z_1 = 5.5$ $Z_2 = 14$	$Z_1 = -1.3$ $Z_2 = 19.6$	$Z_1 = -0.45$ $Z_2 = 18.9$	
4	$Z_1 = 2$ $Z_2 = 13.5$	$Z_1 = -7.6$ $Z_2 = 18.7$	$Z_1 = -6.4$ $Z_2 = 18.05$	
5	$Z_1 = 10.5$ $Z_2 = 16.5$	$Z_1 = 8.5$ $Z_2 = 18.5$	$Z_1 = 8.75$ $Z_2 = 18.25$	

6	$Z_1 = 5$ $Z_2 = 17.5$	$Z_1 = -1.4$ $Z_2 = 20.3$	$Z_1 = -0.6$ $Z_2 = 19.95$	
	$Z_1 = 1.5$ $Z_2 = 17$	$Z_1 = -7.7$ $Z_2 = 19.4$	$Z_1 = -6.55$ $Z_2 = 19.1$	
	$Z_1 = 2.5$ $Z_2 = 20$	$Z_1 = -1.9$ $Z_2 = 20.8$	$Z_1 = -1.35$ $Z_2 = 20.7$	
	$Z_1 = -1$ $Z_2 = 19.5$	$Z_1 = -8.2$ $Z_2 = 19.9$	$Z_1 = -7.3$ $Z_2 = 19.85$	
	$Z_1 = -6.4$ $Z_2 = 20.3$	$Z_1 = -9.3$ $Z_2 = 20.1$	$Z_1 = -8.95$ $Z_2 = 20.15$	

Table 3. Results of example 4 for some cases

Using improved convex combination:

Extreme points are $(0,7), (3,8), (6,7), (9,3)$ and $(10,0)$. Strongly efficient solutions are $(3,8), (6,7), (9,3)$ and $(10,0)$. Positive efficient solutions are

$Z_1(0,7) = 14, Z_1(3,8) = 13,\ Z_1(6,7) = 8,\ Z_1(9,3) = -3, Z_1(10,0) = -10.$

$Z_2(0,7) = 7,\ Z_2(3,8) = 14,\ Z_2(6,7) = 19, Z_2(9,3) = 21, Z_2(10,0) = 20.$

So, they are $(3,8), (6,7)$. Now the linear system of convex combination for $(3,8), (6,7)$ is

$$x_1^* = 3\lambda_1 + 6\lambda_2$$
$$x_2^* = 8\lambda_1 + 7\lambda_2$$
$$\lambda_1 + \lambda_2 = 1.$$

Table 4 shows results for some values of λ_1, λ_2, and comparing with other two methods

λ_1	λ_2	x_1^*	x_2^*	Z_1	Z_2	Z_1 [16]	Z_2 [16]	Z_1 [15]	Z_2 [15]
0.5	0.5	4.5	7.5	10.5	16.5	9.61	17.38	8.3	18.4
0.1	0.9	5.7	7.1	8.5	18.5			Best case	
0.9	0.1	3.3	7.9	12.5	14.5				

Table 4. Result of example 4 with compare

8. Concluding Remarks

A new approach to tackling multi-objective linear programming challenges has been discovered. This method employs the principle of convex combinations. A novel point is generated from the extreme points within the feasible region. The concept relies on new definitions, specifically, strongly efficient and positive efficient. This approach allows for the selection of acceptable solutions for varying values in the convex combination. Numerical examples and an algorithm were provided to clarify the procedure. The effectiveness of these methods was confirmed by comparing the results with other alternative strategies.

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Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Ethical Approval

Ethical approval was not required for this study as it did not involve human participants, personal data.

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طريقة محسنة تعتمد على مجموعة باريتو لحل مسائل البرمجة الخطية ذات الهدف تاو

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الخلاصة: التوليفات المحدبة، وهي فكرة أساسية في الرياضيات، تكتسب أهمية خاصة في الجبر الخطي، والتحليل المحدب، والتحسين. فهي توفر طريقةً لإنشاء نقاط جديدة من مجموعة مُحددة مع الحفاظ على التحدب، وهو سمة أساسية في العديد من المجالات التطبيقية والرياضية. في هذه الورقة البحثية، قمنا بتحسين وتنميط فكرة لحل دالة متعددة الأهداف لإيجاد حل شامل. ركزت الطريقة على استخدام التوليفات المحدبة لبعض النقاط، وتحديدًا النقاط الفعالة الموجبة، وهو تعريف جديد. تُظهر النتائج حلولًا مقبولة لصانع القرار . وبمقارنة هذه الطريقة مع طرق أخرى، توفر هذه الطريقة مجموعة من الحلول، ويمكن استخدامها أيضًا في حالة وجود حلول مثالية فردية على نقاط خارجية مميزة.

الكلمات المفتاحية: وظائف متعددة الأهداف، مزيج محدب، حلول فعالة إيجابية ، حلول فعالة ضعيفة.