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Bayesian Estimation of the Inverse Rayleigh Process under a Non-Homogeneous Poisson Process Framework

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Abstract

The rationale on which this study is based is that accurate and dependable means to obtain time-dependent failure rates in repairable systems, especially in cases that are not homogeneous, are required, and the conventional models are not always in a position to meet these demands. To address this, the research targets at use of Inverse Rayleigh Process (IRP) within a Non-Homogeneous Poisson Process (NHPP) paradigm, a model of the system failures that suits the use of the stochastic model of a system. To improve the accuracy of parameter estimation, the Maximum Likelihood Estimation (MLE) approximation and Bayesian methods are studied and here the solving of the analytical problems due to intractable posterior distributions when using Laplace approximations is sought. Zooming over the simulation experiments that have been conducted on various sample sizes, evaluated through Root Mean Square Error (RMSE), shows that the Bayesian estimator in particular Bayes II prior outperforms MLE. Lastly, the proposed approaches are confirmed on the real-life failure records in the Mosul Gas Power Plant, which confirms the effectiveness of the Bayesian approach in the modeling of the coupled reliability systems in practice and more precisely in the data-scarce context.

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1. Introduction

An analysis of repairable systems requires the use of counting processes to track system failures through individual events. The analysis of repairable system reliability requires an investigation of multiple repair interventions throughout its operational lifespan. When failure data shows no discernible pattern the renewal process serves as an adequate simulation model because system repairs bring the system back to pristine condition [1],[2]. The decision to use the Inverse Rayleigh Process (IRP) as the model framework is guided by the flexibility of this type of process that successfully models the behavior of systems which have the tendency or the character of early-life failures or reliability growth profiles. The IRP is analytically tractable (i.e. does not require numerical methods) and is a member of the exponential family; unlike the traditional functions, such as exponential and Weibull, it is unimodal in probability density function, and is therefore particularly appropriate to modeling repairable systems in non-homogeneous conditions. It takes into account naturally the reduction in the failure rates over time that is characteristic of the reliability data of systems subject to either corrective or preventive maintenance because the intensity function naturally admits such a dependency. The above features make the IRP an attractive prospect to alternative models based on renewal or NHPP, especially those interested in improved prediction and more stable estimates in practice. The research aims to determine the Inverse Gompertz process parameter estimates which handle time-dependent failure rate changes correctly. The assessment involves two parameter estimation approaches which include Maximum Likelihood Estimation (MLE) and Bayesian inference (Bayes method). Both approaches deliver inference methods that operate through frequentist statistics and probabilistic theories to determine process parameters. Reliability growth testing requires assessments to identify if systems exhibit meaningful changes either positive or negative throughout the real-time duration. Interfresh periods that rise during the testing phase point to better system reliability between failures. System reliability tends to deteriorate when the intervals between system failures become shorter. The Non-Homogeneous Poisson Process (NHPP) stands out for modeling dynamic failure patterns because its intensity function allows modeling changes in failure rates throughout time. The failure intensity rate decreases throughout time but it increases as failures occur more frequently. Renewal processes feature inter-failure times that both have the same distribution type and independence of each other element. A renewal process becomes a Homogeneous Poisson Process when each inter-arrival time follows an exponential distribution pattern according to [3],[4]. The authors advance reliability research through thorough assessment of MLE and Bayesian estimators for Inverse Gompertz processes that support trend-based maintenance plans and failure predictions.

A number of researches have considered different estimation methodologies associated with problems of fuzzy reliability analysis and its statistical counterparts. a statistical model based on frailty of repairable systems with dependent failure times was developed looking at perfect repairs. They calculated parameter estimation procedures, carried out simulation experiments and compared their model with practical data of sugarcane harvesters and dump trucks. They aimed to get to know more and measure the unobserved heterogeneity and dependence of system failures better to enhance the analysis of reliability and that of maintenance strategies. They also commented on future research options, which include enlarging the model to incomplete fixes and Bayesian methods [5]. Presents a Bayesian framework for change-point detection in non-homogeneous Poisson processes (NHPP) with a Weibull (power-law) intensity function, applying MCMC and model selection via Bayes factor and DIC closely aligning [6]. Developed and validated a hybrid survival model using Burr Type XII distribution combined with MLE and SVM methods to predict breast and brain cancer survival times [7]. In addition, the Non-Homogeneous Poisson Process (NHPP) has been used in telecommunications so as to model the arrival rates of calls and messages. In a novel contribution, researchers suggested an NHPP model, where the survival rate of the patients is following an inverse Gompertz distribution under fuzzy data conditions. Both the classical and smart (AI-based) methodologies of estimating were used to carry out the estimations of the model parameters. The results of these methods were compared in the fuzzy as well as the real data settings – the aim was to determine the method with the most adequate estimation method [8–12]. Having put the model of repairable systems situated in the context of motivation and theoretical basis, i.e., using non-homogeneous models of stochastic frameworks, notably the Inverse Rayleigh Process (IRP), the next point is to formally elaborate on the proposed model building. These consist of the probabilistic framework, intensity functions and related mean value functions, which characterize the IRP within NHPP context. In laying down these mathematical foundations, a rigorous evaluation of the estimation techniques of parameters will be provided which is fundamental to the practical application and capacity to predict of the suggested model.

1.1. Inverse Rayleigh Process (Proposed Model)

Flexible and analytically treatable is Inverse Rayleigh (IR) distribution that has attracted much attention in studies about life-testing and reliability analysis. Its suitability is especially evident in describing the time-dependent rate of events subject to the nonhomogeneous Poisson processes and thus, as a result, giving rise to the Inverse Rayleigh process. With a unimodal pdf and belonging to the exponential family, the IR distribution has excellent modeling and forecasting potential for complex system failures rates. Based on its mathematical properties and interpretability, it is a useful tool for informed decision making in such industries as manufacturing, engineering, and healthcare [13].

$$f(t) = \lambda(t)e^{-m(t_0)}. \qquad 0 < t < \infty \tag{1}$$

Where $f(t)$ is the probability density function of the time until the first failure/event occurs, evaluated at time t, $\lambda(t)$ is the intensity function of the NHPP at time t, indicating the instantaneous rate at which events (e.g., failures) are expected to occur. And $m(t)$ is the mean value function (MVF) evaluated at time t_0, representing the expected cumulative number of events from time 0 up to time t_0.

Proposed the time rate of occurrence, denoted as $\lambda(t)$, in the new process is defined by the equation [14],[15]:

$$\lambda(t) = \frac{2\alpha^2}{t^3} \quad 0 < t < \infty \ , \ \alpha > 0, \tag{2}$$

Where t represents time, and the parameters α is positive constants. The mean value function is expressed as follows:

$$m(t) = \int_0^t \lambda(u)\, du \qquad , \; 0 < t < \infty \tag{3}$$

$$= \int_0^t \frac{2\alpha^2}{u^3}\, du$$

$$= -\alpha^2\, t^{-2}\, .\; 0 \le t \le t_0 \tag{4}$$

We will substitute equations (2) and (4) into equation (1), The Inverse Rayleigh process is obtained through the pdf, denoted by [16],[17]:

$$f(t) = \frac{2\alpha^2}{t^3}\, e^{-\alpha^2 t_0{}^{-2}} \quad , \qquad t > 0\, . \tag{5}$$

In this equation, α controls the shape of the inverse Rayleigh intensity, acting as a scale parameter. It is important to correctly estimate α, since it shapes how well the model works and how it describes the data. Proper calculation of this parameter helps the Inverse Rayleigh Process perform more successfully in studying reliability. Having determined the IRP model structure, one is now interested in the statistical estimation of the parameters of the model. This is a decisive measure, which helps to determine the applicability of this model because parameters estimation determines the accuracy of system failures predictions. Two chief estimation paradigm regimes are viewed: the old fashioned Maximum Likelihood Estimation (MLE) and the Bayesian framework of inference.

2. Method of Estimation

There are several methodological techniques that can be used in estimating the parameters of Inverse Rayleigh process. In the research reported, a comparative study was carried out in which both the techniques of Maximum Likelihood Estimation (MLE) as well as Bayesian estimation techniques were used to assess their performance and inferential properties.

2.1. Maximum likelihood Method (MLE)

Maximum Likelihood Estimation (MLE) is a fundamental technique and a widely used parameter estimator in stochastic models' formulations. Its prevalent use is mainly owed to desirable statistical property, namely, its consistency, asymptotic unbiasedness and efficiency under regularity conditions. In this sense, the main goal of MLE is to find that set of the parameter values that maximizes the likelihood function on the basis of the observed data, so that the estimates obtained are as good as possible in terms of explaining the probabilistic mechanism underlying the data. In the event of a Non-Homogeneous Poisson Process (NHPP) in which the intensity function for the time-dependent rate of occurrence is given by Equation (5), the joint probability density function for observed event times is $(t_1, t_2, \dots, t_n)$ is formally expressed by the following equation [18],[19]:

$$f(t_1, t_2, \dots, t_n) = \prod_{i=1}^{n} \lambda(t_i) e^{-m(t_0)} \tag{6}$$

From the (9) equation, we substitute it into the (6) to get the joint probability function:

$$f(t_1, t_2, \dots, t_n) = \prod_{i=1}^{n} \frac{2\alpha^2}{t_i{}^3}\, e^{-\alpha^2 t_0{}^{-2}} \tag{7}$$

The Likelihood function for the formula (7) for the period (0, t].

$$L = \prod_{i=1}^{n} \frac{2\alpha^2}{t_i^3}\, e^{-\alpha^2 t_0^{-2}} \tag{8}$$

The log-likelihood function is expressed as follows:

$$\ln L = n \ln(2\alpha^2) - 3 \sum_{i=1}^{n} \ln(t_i) - n\, \alpha^2\, t_0^{-2} \tag{9}$$

Hence, deriving equation (9) with respect to parameter α, we get:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{2\alpha^2} \cdot 4\alpha - 2\, n\, \alpha\, t_0^{-2} \tag{10}$$

And formula (10) is equating to zero, the likelihood will be:

$\frac{n}{2\alpha^2} \cdot 4\alpha - 2\, n\, \alpha\, t_0^{-2} = 0$ (11)

Therefore, the maximum likelihood estimator for the parameter α is:

$\widehat{\alpha}_{MLE} = \sqrt{t_0^2}$ (12)

Where t_0 represent the time, the last event occurred.

2.2. Bayes Estimation (Bay)

Bayesian estimation is commonly known as one of the most successful methodologies of inferring the parameters of stochastic processes because of its natural capability of providing reliable posterior inferences incorporating prior knowledge. This technique presupposes a fundamental precondition of the prior distribution of parameters according to which the existing beliefs or empirical knowledge are contained before the observation of the current data. It then combines this prior with the likelihood function, i.e., one derived from the sample data being observed, and expressed through maximum likelihood estimation; to arrive at the posterior distribution, which is a representation of the updated views of the parameters, in light of the data provided. In view of the likelihood function that is given in Equation (9), we assume that prior distributions for all parameters follow Gamma distribution, which is corroborated by the existing studies [20],[21].

$\alpha \sim Gamma(a, b)$. The $p.d.f$ for each parameter are:

$$p(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha} \quad (13)$$

and the joint prior distribution function for (α) is [13][14]:

$$p(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha} \quad (14)$$

then the posterior distribution function is:

$$p(\beta, \lambda | Data) = p(\beta, \lambda) L(\beta, \lambda)$$

$$= \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha} \prod_{i=1}^{n} \frac{2\alpha^2}{t_i^3} e^{-\alpha^2 t_0^{-2}}$$

$$= \frac{b^a}{\Gamma(a)} \alpha^{a+2n-1} e^{-(b\alpha + n\alpha t_0)}$$

$$= \frac{\alpha^{a+2n-1} e^{-(b\alpha + n\alpha t_0)}}{\int \alpha^{a+2n-1} e^{-(b\alpha + n\alpha t_0)} d\alpha} \quad (15)$$

The Bayes estimator for IRP parameters can be obtained as follows:

$$\alpha_{Bay} = E[\alpha | Data] = \int p(\alpha | Data) d\alpha = \frac{\int \beta^{a+n} \lambda^{c+n-1} e^{-(b\beta + d\lambda) - \lambda t_0^{\beta}} \prod_{i=1}^{n} t_i^{\beta-1} d\beta}{\int \alpha^{a+2n-1} e^{-(b\alpha + n\alpha t_0)} d\alpha} \quad (16)$$

Because of the analytical intractability of the integrals in Equation (16), Laplace approximation was used as a pragmatic and easy approach to approximating the posterior expectations. This approach is especially helpful if closed-form solutions lack and also it allows achieving a good approximation in Bayesian inference in the case where conditions are regular [15][16][17],[22],[23]. Consider the following general formulation:

$$I(Data) = E[u(\alpha|Data)] = \frac{\int e^{(ln[u(\alpha)]+\ell+\rho)}d\alpha}{\int e^{(\ell+\rho)}d\alpha} \quad (17)$$

In this case, the function $u(\alpha)$ stands for a function of α. for the case of the Equation (16), it is particularly regarded as the α itself. This function is the natural logarithm of the prior probability distribution of the parameter, and it is given formally as below:

$$\rho = ln\left[\frac{b^a}{\Gamma(a)}\right] + (a-1)ln(\alpha) - (b\alpha) \quad (18)$$

ℓ is the natural logarithm of the likelihood function, defined as follows:

$$\ell = nln(\alpha) - t_0^{\alpha} + (\alpha - 1)\sum_{i=1}^{n} ln(t_i) \quad (19)$$

Let:

$$h(\alpha) = \frac{1}{n}(\ell + \rho) \quad (20)$$

$$h^*(\alpha) = \frac{1}{n}ln\big(u(\alpha)\big) + h(\alpha) \quad (21)$$

Then the equation (17) becomes:

$$I(Data) = E[u(\alpha|Data)] = \frac{\int e^{nh^*(\alpha)}d\alpha}{\int e^{nh(\alpha)}d\alpha} \quad (22)$$

Thus, Laplace's estimate for this equation is as follows:

$$I(Data) = E[u(\alpha|Data)]$$

$$= \left[\frac{|\Sigma^*|}{|\Sigma|}\right]^{1/2} exp\{n\big(h^*(\alpha^*) - h(\alpha)\big)\} \quad (23)$$

The values that maximize the function $h^*(\alpha^*)$, and (α) are the values that maximize the function $h(\alpha)$, Σ^* and Σ are the negative inverse of the Hessian Matrix for h^* (α^*) and $h(\alpha)$ at (α^*), and (α) respectively:

$$\Sigma = -\frac{\partial^2 h}{\partial \alpha^2} \quad (24)$$

$$\Sigma^* = -\frac{\partial^2 h^*}{\partial \beta^2} \quad (25)$$

Note that h is a constant, while h^* change with u, whereas:

$$h^*{}_\beta(\alpha^*) = \frac{1}{n}ln(\alpha) + h(\alpha) \quad (26)$$

Hence, the Bayes estimators for the parameters of the Inverse Rayleigh Process (IRP) found with the use of Laplace approximation approach are as follows:

$$\hat{\alpha}_{B.LC} = \left[\frac{|\Sigma^*|}{|\Sigma|}\right]^{1/2} exp\{n\big(h^*{}_\alpha(\alpha^*) - h(\alpha)\big)\} \quad (27)$$

In order to compare and point out the most effective estimation method the criterion of Root Mean Square Error (RMSE) was used and calculated as follows [18]:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{Q}(\widehat{m(t)}_i - m(t))^2}{Q}} \tag{28}$$

where $m(t)$: represents the real value, $\widehat{m}(t)$: represents the estimated value. Since the MLE and Bayesian methods have been introduced to estimate IRP parameters, it is quite important to evaluate the empirical performance of the developed methods. The next part shows the simulation study aimed at comparing the estimators under different sample sizes and parameter conditions. This is essential towards a confirmation of the soundness and comparative accuracy of the methods of estimation.

3. Simulation

Simulation has a central role in system and process study and analysis. in which real or imaginary operations are defined mathematically or computationally. Being a highly flexible instrument of analysis, simulation allows researchers to study the impact of the different system parameters and to explore hypothesis-driven simulations without the limitations of physical experimentation. In numerous of the day-to-day applications, empirical testing may be prohibitively costly, in desirable or time-consuming; in such a case simulation would become an essential alternative. Through purposeful changes of input parameters and performing a series of controlled virtual experiments, simulation enables a deeper understanding of system dynamics and validation of theoretical models in support of better decision making in science and practice in various disciplines engineering, economics and software systems [17].

Stage I: Model Initialization and Parameter Specification

The first step plays a pivotal role by creating a base for all following simulation processes. The first step includes all operations that establish core hypothesis along with parameter value selection while defining process behavior. This phase contains three sequential elements for completion:

Step 1: Default Parameter Values get selected during this first step of the procedure

The simulation process starts by setting initial default values to the parameters used in Inverse Rayleigh Process. The chosen parameter settings draw from past experimental studies together with comprehensive testing work to maintain robustness and applicability of configured parameters. Two specified parameter configurations showed the best results from evaluating different simulation parameter options.

- **Set 1:** $\alpha = 0.6$; $\alpha = 1.6$

These parameters respectively define the shape, scale, location, and additional distributional characteristics necessary for generating synthetic data that closely resemble the theoretical behavior of the Inverse Rayleigh Distribution. As shown in the table 1.

Step 2: Determination of Sample Sizes

Different sample sizes of small medium and large datasets successfully measure the stability and performance of the estimators during the simulation.

- n =25;50;100.

This stratification allows for rigorous analysis of estimator sensitivity and efficiency under varying data volumes.

Stage II: Random Data Generation via Inverse Transformation

This stage involves the generation of pseudo-random data points that follow the probability distribution function of the Inverse Rayleigh Process, utilizing the Inverse Transform Sampling Method.

Step 1. Generation of Uniform Random Variables

Let $u_i \sim U(0,1), i = 0,1,2, \dots, n.$ (29)

MATLAB provides the built-in rand function to produce independent identical distributed $(i.i.d.)$ random variables distributed uniformly from the interval (0,1) during this stage. The formal expression is:

Where:

- u_i: Continuous uniform random variable.
- n: Sample size.

Step 2: Transformation to Inverse Rayleigh Distribution Data

The generated uniform variables are transformed into data that follow the Inverse Rayleigh Process via the Inverse Cumulative Distribution Function (CDF). This transformation leverages the known CDF of the Inverse Rayleigh Distribution, denoted as Eq. (1) in the study, and applies the inverse mapping: $x_i = F^{-1}(y)$, This simplifies to:

$$t_i = \sqrt{\frac{u}{-\alpha^2}}, i = 0,1,2,\dots,n. \tag{30}$$

This procedure ensures that the synthetic dataset accurately represents the statistical characteristics of the Inverse Rayleigh Process under study.

Stage III: Parameter Estimation

The simulation framework advances to its last stage through parameter estimation of Inverse Rayleigh distribution as applied to Software Reliability Growth Models (SRGMs). The third phase includes multiple technical approaches for parameter estimation across the complete observation period to guarantee predictive reliability and statistical precision. These estimation methodologies are used for the process:

- Maximum Likelihood Estimations.

- Bayes estimator.

Stage IV: The optimal estimation method was identified based on the comparison metric Root Mean Squared Error (RMSE), evaluated across the estimation of the probability density function.

Stage V: experiment is repeated (1000) times.

Stage VI: Compute the Root Mean Square Error (RMSE) for each observation t_i, based on the estimated distribution parameters c and k.

$$RMS(\hat{a}) = \sqrt{\frac{\sum_{i=1}^{Q}(\widehat{\alpha_i} - \alpha_i)^2}{Q}}, \tag{31}$$

Stage VII: Schwarz Information Criterion (SIC/BIC): to compare models:

Compared to the Root Mean Square Error (RMSE), this paper adds the Schwarz Information Criterion (SIC), as also known as the Bayesian Information Criterion (BIC), to gauge model performance through penalization of model complexity. The BIC of this is calculated as [25]:

$$BIC = -2\log L(\hat{\theta}) + k\log(n), \tag{32}$$

Where $L(\hat{\theta})$ represents the maximum likelihood of the model and k is the number of free parameters, n is the sample size. The criteria is a tradeoff between model fit and parsimony. In the case of both the Maximum Likelihood Estimation (MLE) as well as the Bayesian estimators, there was the computation of BIC based on both simulated as well as real data. The model with the least BIC is taken as the most desirable and provides the best trade-off between the explanatory power and complexity.

Table 1. Specification of default values of a parameter for the prior distribution in Bayesian estimate.

Case	a
I	0.6
II	1.6
III	0.6
IV	1.6

Table 2. The simulated $RMSE$ for the MLE and Bayes estimator for IRP.

n	parameter	MLE	Bay Ⅰ	Bay Ⅱ	Bay Ⅲ	Bay Ⅳ
25	$\alpha = 0.6$	29.397	28.731	27.569*	27.674	28.879
	$\alpha = 1.6$	29.298	21.888	21.563*	27.356	24.161
50	$\alpha = 0.6$	18.914	18.687	18.321*	18.361	18.757
	$\alpha = 1.6$	18.385	15.594	15.041*	17.097	16.676
100	$\alpha = 0.6$	13.061	12.989	12.853*	12.875	13.005

	$\alpha = 1.6$	10.844	10.097	10.093*	11.006	10.319

Table 3. The simulated BIC for the MLE and Bayes estimator for IRP.

n	parameter	MLE	Bay Ⅰ	Bay Ⅱ	Bay Ⅲ	Bay Ⅳ
25	$\alpha = 0.6$	19.287	27.631	16.469*	26.774	26.679
	$\alpha = 1.6$	19.198	20.788	11.363*	26.456	22.061
50	$\alpha = 0.6$	28.814	17.587	17.211*	19.461	16.557
	$\alpha = 1.6$	28.495	14.494	14.031*	18.197	14.476
100	$\alpha = 0.6$	23.051	11.889	11.753*	13.975	11.905
	$\alpha = 1.6$	20.734	9.097	9.083*	12.106	12.119

The numerical results shown in Tables 2 and 3 are indicative of the estimates of the model parameters of the Inverse Rayleigh Process (IRP) that was estimated using the Maximum Likelihood Estimation (MLE) and other Bayesian estimation techniques. A comparison of the models using the Root Mean Square Error (RMSE) and Schwarz Information Criterion (SIC), values shows that the Bayes II is performing better than other Bayesian techniques and MLE method for better estimation accuracy under the stated evaluation criteria. Although simulations give a hint at how things are to be done under controlled conditions, real time validation is essential. In this way, the next section is devoted to the application of suggested estimation approaches to the real failures data of the Mosul Gas Power Plant. This application does not only prove the practicality of the model but also proves its applicability in supporting operational conditions.

4. Simulation

In order to determine the real-world applicability of the proposed estimation methods, Iowa State University's failure data collected from Mosul Gas Power Plant, placed in Nineveh Governorate of Iraq were used. The dataset consists of observed lags (days) between the consecutive failures observed during the period May 1st, 2019 – June 30th, 2021.

4.1. Homogeneity Testing for the Inverse Rayleigh Process

The IRP is considered as nonhomogeneous because its time rate of events is dependent on the change in time (t), which means that its behavior is affected by time t. Therefore, the Inverse Rayleigh process is homogeneous when $\lambda = 0$, and it is nonhomogeneous when $\lambda \neq 0$. To test whether the process is homogeneous or nonhomogeneous, the following hypothesis is considered [17]:

$H_0: \lambda = 0$

$H_1: \lambda \neq 0$

which can be tested through the following statistics:

$$Z = \frac{\sum_{i=1}^{n} \tau_i - \frac{1}{2} n\tau_0}{\sqrt{\frac{n{\tau_0}^2}{12}}} \tag{33}$$

Where:

Z represent calculate test.

$\sum_{i=1}^{n} \tau_i$ is the sum of the accident times for a period $(0, \tau_0]$,

n represents the number of accidents that occur in a period $(0, \tau_0]$.

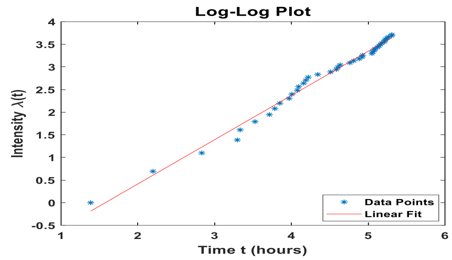

Figure 1. Cumulative number of days of operation between two shutdowns with their occurrence times on a logarithmic scale.

The scatter plot shows evident linear tendency, which implies that the data is highly suitable to the modeling by means of the Inverse Rayleigh Process (IRP) function. Similar performance of the Maximum Likelihood Estimation (MLE) and Bayesian Estimation (Bayes) especially in a sample size of 50 could be referred to as the strengths on which the two methodologies are based on. Both also attempt to reduce the gap between observed and predicted values, but they do so in different inferential structures MLE finds the value of the parameters that maximizes likelihood of the data given a set of (fixed) parameters, whereas Bayes use prior information to update a given (prior) distribution to a posterior using the data. This agreement in the performance demonstrates the strength of the performance of the LRP model at moderate sample size.

4.2. Homogeneity Testing for the Inverse Rayleigh Process

To test the homogeneity of the data under study, we used the statistical laboratory in formula (32), with a MATLAB/R2019a program specifically designed for this purpose. The calculated value of |Z| was found to be 64.2496, which is higher than its corresponding tabular value of 1.96 at a significance level of 0.05. Therefore, we reject the null hypothesis and accept the alternative hypothesis. This indicates that the process under study is nonhomogeneous.

Table 4. $RMSE$ values for methods used to estimate the IRP parameters

Unite	Size	Method	$\hat{\alpha}$	RMSE	BIC
M1	49	MLE	0.31268	26.948	24.726
		Bay I	0.36394	22.906	20.716
		Bay II	0.44898	**18.303***	**16.103***
		Bay II	0.45423	38.306	36.106
		Bay IV	0.36995	43.019	41.018
M2	50	MLE	0.2248	29.985	27.775
		Bay I	0.3870	27.576	25.476
		Bay II	0.4637	**23.703***	**21.503***
		Bay II	0.4682	24.041	22.031
		Bay IV	0.3827	27.308	25.108
M3	50	MLE	0.30016	25.439	23.239
		Bay I	0.36025	22.673	20.473
		Bay II	0.43591	**19.322***	**17.122***
		Bay II	0.43956	19.62	17.61

Bay IV	0.35682	22.559	20.359

Table 4 shows estimate of parameters of Inverse Rayleigh Process (IRP) computed from the proposed estimation schemes of this study. From the values of the Root Mean Square Error (RMSE), Bayes II model performed better as it provided more efficient estimators in describing the underlying structure in data. The figures following below illustrate the IRP as estimated by each technique, compared to the actual cumulative failure intervals realized at the power station hence demonstrating the relative precision of techniques of estimation. Figures are provided here that estimate the cumulative failure functions using both Maximum Likelihood and Bayesian methods and they are compared to the observed cumulative failure data for each station. The results show that both models can match actual behavior, but on all stations, Bayes II fits better than the open-source procedure.

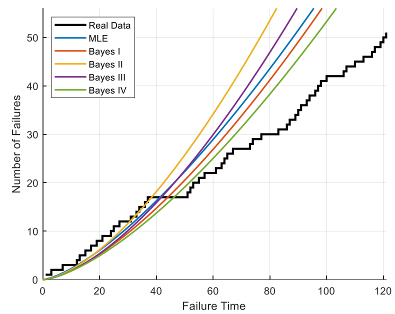

Figure 2. Comparative Estimation of the Cumulative Failure Function for Station M1 Based on MLE and Bayesian Methods under the Inverse Rayleigh Process Model.

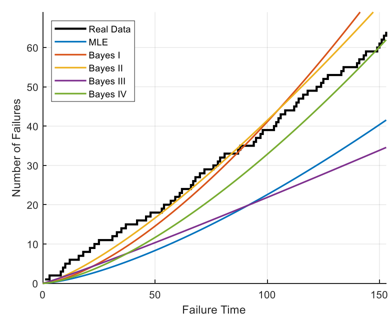

Figure 3. Parsimonious model fitting of the cumulative failure function for Station M2 employing Maximum Likelihood and Bayesian estimators based on the Inverse Rayleigh process.

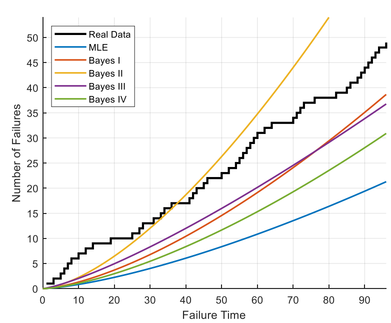

Figure 4. Comparative Estimation of the Cumulative Failure Function for Station M3 Based on MLE and Bayesian Methods under the Inverse Rayleigh Process Model.

Figures above display the estimated failure functions for the Mosul Gas Power Plant with different estimation procedures. It is clear from comparing the models that the Bayesian technique, especially the Bayes II model, is closest to fitting the data. Therefore, the Bayesian method gives more accurate and efficient results compared to other estimation strategies when trying to reveal the way failures happen. The measured conclusion that is supported by real-life data proves the theoretical and simulation-based results, especially focusing on the effectiveness of Bayes II estimator. The last section concludes the findings, summarizes them with the available evidence based on both synthetic and actual datasets, and discusses how to apply the current work to the domain of reliability modeling and future methodological advancement, as well as interdisciplinary application.

5. Conclusion and Future Work

The study comes up with a way to fit the Inverse Rayleigh Process into a Non-Homogeneous Poisson Process (NHPP) and applies maximum likelihood estimation and Bayesian inference. Results from simulations and inspection of real failures at the Mosul Gas Power Plant prove that using the Bayes II method makes the parameter estimates more reliable and accurate than with classical statistics. In the simulation studies, Bayes II was always better than the other estimators at providing accurate and stable estimations. The Bayes II model was found to be the superior choice for modeling the failure behaviors of the repairable system in the power plant study. By performing homogeneity testing, it was confirmed that the IRP was not homogeneous and modeling with a NHPP was justified. Furthermore, the use of the Laplace approximation allowed for doing Bayesian inference in an efficient way, despite the fact that the posterior distributions were not easy to evaluate. All in all, the study demonstrates that Bayesian techniques are especially useful when there is available information, in dealing with failure rates that change over time in complex systems. Moving forward, researchers could widen the reach of this Bayesian model to handle failures involving more than one attribute by using hierarchical or copula-based models. It is also possible to perform real-time estimation using moving priors with the help of particle filters. Additionally, nonparametric Bayesian techniques such as those based on Dirichlet Process priors, can allow for further flexibility. Looking at the results of machine learning models like neural networks and ensembles might reveal more about financial conditions when there is enough data. A next step would be to investigate the uses and success of IRP-based NHPP modeling in diverse areas such as aerospace, telecommunications and healthcare. In addition to being statistically interesting, the results are of great practical value in the engineering and industrial reliability analysis worthy of study. A correct degree of modeling failure behavior with the Inverse Rayleigh Process allows maintenance engineers and reliability managers to better time actions of preventive maintenance, minimize occurrence of unplanned downtimes, and maximize use of resources. This is especially important with fields with mission-critical reliability like generation of power, manufacturing and aerospace systems. The superiority in estimation displayed by the Bayesian estimator particularly in situations of limited data makes it possible to state that even when the organizations operate on reoccurring or speculative failure records, they would still be able to generate sound reliability estimates supporting decision-making through data analytics in the field.

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التقدير البايزي لعملية رايلي العكسية في إطار عملية بواسون غير المتجانسة

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الخلاصة: الأساس الذي تستند إليه هذه الدراسة هو الحاجة إلى وسائل دقيقة وموثوقة للحصول على معدلات الفشل المعتمدة على الزمن في الأنظمة القابلة للإصلاح، وخاصة في الحالات غير المتجانسة، حيث لا تكون النماذج التقليدية دائمًا قادرة على تلبية هذه المتطلبات. ولمعالجة هذا الأمر، تستهدف الدراسة استخدام عملية رايلي العكسية (IRP) ضمن إطار عملية بواسون غير المتجانسة (NHPP)، كنموذج لفشل النظام يتماشى مع استخدام النموذج العشوائي للنظام. ولتحسين دقة تقدير المعلمات، يتم دراسة طريقتي التقدير بالاحتمالية العظمى (MLE) والطريقة البايزية، حيث يتم البحث في حل المشكلات التحليلية الناتجة عن صعوبة التعامل مع التوزيعات اللاحقة باستخدام تقريبات لابلاس. وتُظهر التجارب العددية التي أُجريت على أحجام عينات مختلفة، والمُقيمة باستخدام متوسط الجذر التربيعي للخطأ(RMSE) ، أن المُقدر البايزي، وخاصة باستخدام التوزيع المسبقBayes II ، يتفوق على .MLE وأخيرًا، تم تأكيد فاعلية الأساليب المقترحة باستخدام بيانات حقيقية عن حالات الفشل في محطة الموصل لتوليد الطاقة الغازية، مما يُثبت كفاءة الطريقة البايزية في نمذجة أنظمة الموثوقية المترابطة في التطبيق العملي، وبشكل أدق في السياقات التي تعاني من شُح البيانات.

الكلمات المفتاحية: عملية رايلي العكسية؛ اختبار التجانس؛ المُقدِّر البايزي؛ المحاكاة.