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A New Theorem for Lower Bounds in NP-Hard Multi-Objective Scheduling Problems

Hassan A. Mahmood[1] **Ayad M. Ramadan**[2]

[1]Department of Mathematics, College of Education, Garmian University, KRG, Iraq,[2]Department of Mathematics, College of Science, Sulaimani University, KRG, Iraq.

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Correspondence:
Hassan A. Mahmood
hassan.aziz@garmian.edu.krd
ayad.ramadan@univsul.edu.iq

Abstract

On a single machine, each of n jobs must be processed continuously. At time zero, every job is ready for processing. The tasks to process a sequence that minimizes the total sum of competition times plus the sum of tardiness $(\sum_{i\in N} c_i + \sum_{i\in N} Ti)$. This bi-criteria problem is NP-hard because of the second one. We provide a theorem that demonstrates a relationship between the optimal solution, lower bounds, and the number of efficient solutions. The case is that the theorem works for NP-hard problems, whereas in previous works the focus was on P-hard problems. The theorem limits the lower bound's range, which is crucial for determining the best answer. Additionally, the theorem allows for discovering new lower bounds by opening algebraic procedures and concepts.

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1. Introduction

Over the past few decades, multi-objective optimization has gained importance as a study area because it has been shown to be a simple and successful method for solving practical optimization problems (Manal and Faez, 2020). Multi-objective optimization's conflict restricts the scheduling direction of workable solutions, and current algorithms struggle to balance parameters or fail to identify the best solution for the problem (Liu et al., 2020, 2023). There are many papers addressing functions with many objectives in scheduling (Lauff and Werner, 2004). (Qader et al. 2023) focused of some structures on multi-objective fuzzy scheduling problems. (Salh and Ramadan, 2022) introduced three algorithms for bi-objective problems in hierarchical and simultaneous cases. Also, many studies found for flow shop problems. (Abdulkareem Zeidan, 2013) proposed genetic algorithm to finding the optimal schedule with minimum makespan. The following the issue class is taken into consideration: n jobs $1,2,\dots,n$ must be handled by one machine (m=1). which becomes accessible at time zero, requires processing time that is positive p_i (Manal and Faez, 2022). A processing time is assigned to each job p_i and a due date d_i, are detailed. With a timetable, we can calculate the tardiness The completion time ci for every job i, and $T_i = \max\{c_i - d_i, 0\}$ and $T_{\max} = \max\{T_i\}$. In this paper the sum of completion times is a multi-objective function that we examine.$\sum_{i\in N} c_i$ added with the sum of tardiness $\sum_{i\in N} Ti$ on single machine. A theorem is presented to find a range of lower bounds in such cases when one of the criteria is NP-hard namely $\sum_{i\in N} Ti$ (Du and Leung, 1990).

2. Related Works

Nowadays, two to three optimization objectives are taken into consideration in the majority of multi-objective optimization situations. Scholars mostly employ, heuristic algorithms, hybrid algorithms, etc. to address these many issues. The multi-objective under consideration is the sum of completion times $\sum_{i\in N} c_i$ added with the sum of tardiness $\sum_{i\in N} Ti$ on single machine. There are two directions of research regarding multi-criteria.

First, researchers using branch and bound with some new lower bounds for multi-objective problems. (Abdullah, 2010) presented algorithms to solve primary and secondary criteria. (Abdul-Razaq and Ali, 2015) used branch and bound algorithm to minimize three objectives simultaneously. (Chachan, and Hameed, 2019) applied four upper bounds to solve four criteria by branch and bound algorithm. (Chachan and Ali, 2020) discussed five criteria problem, and used branch and bound algorithm. (Ibrahim et al., 2022) reported two methods to solve five criteria problems namely branch and bound, and local methods. Lastly, (Zhao, Q., Yuan,2024) showed an interesting structure of lexicographical type of problem, they studied the total tardiness as primary criterion where maximum lateness, weighted number of tardy jobs, and the total weighted tardiness were secondary criteria.

Second, some theories were presented to restrict lower bounds for multi-objective functions. These techniques help the decision makers to choose the best solution among a set of comprised solutions. (Ramadhan and Jabbar,2006) formulated a new theorem for total completion times plus the maximum tardiness.

To restrict the range of lower bounds. (Ramadan, 2011) generalized the idea to sum of weighted completion times added with maximum weighted tardiness. (Amin and Ramadan, 2021) studied the sum of maximum earliness and maximum tardiness by the sum technique. (Mahmod et al., 2022) applied the theorem for the total square completion time and maximizing earliness. In fuzzy environment with triangular fuzzy numbers, (Ramadan, 2021, 2023) extended the idea to the total fuzzy completion time and maximum fuzzy earliness problem, and defined a new definition which is m-strongly positive fuzzy numbers also, minimizes total fuzzy completion time and maximum fuzzy tardiness. Forn tri-criteria, (Hassan et al., 2022) generalized the idea to three-criteria problem of minimize the sum of total completion time, maximum earliness and maximum tardiness using efficient solutions. Also, (Sharif and Ramadan, 2023) studied three criteria of maximum earliness, maximum tardiness, and maximum late work simultaneously

Focusing to these criteria, the criterion is solvable in polynomial times (P-hard). That is why the theorem works effectively. In this paper, and for the first time we introduce the same idea for an NP-hard criterion $\sum_{i\in N} Ti$ which is one of the difficult criteria in scheduling (Zhao, Q., Yuan,2024).

3. Main Problem

Since $\sum_{i\in N} Ti$ is NP-hard problem, the theorem fails to determine the range of lower bound. So, we present a condition on problems that related to the main problem.

4. Job- Scheduling Concepts

Start with some important notations of scheduling problems. Here, we only focus on the used notations on single machine. Jobs i, (i = 1, …, n) have (Manal and Faez, 2020):

N =the set $\{1,2,3, \dots, n\}$.
P_i = processing time for job i .
d_i = due date for job i .
C_i = completion time for job i.
L_i = lateness of job i, $L_i = C_i - d_i$
T_i = tardiness of job i, $T_i = \max\{L_i, 0\}$
EDD- rule: (early due date) The order of the jobs is nondecreasing of d_i
SPT-rule: (short processing time) The order of the jobs is nondecreasing of p_i.
LB: (lower bound) It is below or equal to the optimal value.
UB: (upper bound) It exceeds or is equivalent to the optimal value.
Opt.: optimal solution.

5. Relation between Optimal and Efficient Solutions:

Our objective function is

$$\sum_{i\in N} C_i + \sum_{i\in N} T_i. \quad (1)$$

Before giving the relationship between efficient solutions and the optimum value, or LB we define the following
LB = $\sum_{i\in N} C_i$(SPT) + $T_{\max}(opt.)$
UB = $\sum_{i\in N} C_i$(SPT) + $\sum_{i\in N} T_i$ (SPT)

Theorem:
A non-negative integer r exists such that $LB + r$ = optimal value, $r \in [N_1 - 1, N_2^* + 1]$ iff $\sum_{j=0}^{k-1} \sum_{\substack{i=2 \\ i>j+1}}^{k-1} N_i$-(j+1) ≥ i-1-j, where N_1 = number of efficient solutions of (1), $N_2^* = \sum_{i\in N} T_i\,(opt.) - T_{\max}\,(opt.)$.

Proof:
Since $LB \le$ optimal value, Thus, a non-negative integer r exists such that LB + r = optimal value satisfies the theorem's first component. It is still to be proven that $r \in [N_1 - 1, N_2^* + 1]$ or to show $N_1 - 1 \le r \le N_2^* + 1$. Now LB+r = optimal value, thus r = optimal value- LB ≤ UB - LB $= \sum_{i\in N} c_i(SPT) + \sum_{i\in N} T_i(SPT) - \sum_{i\in N} c_i(SPT) - T_{\max}(opt.) = \sum_{i\in N} T_i(opt.) - T_{\max}(opt.) = N_2^* \le N_2^* + 1$. Hence $r \le N_2^* + 1$. We will prove $N_1 - 1 \le r$ by induction on N_1.
If $N_1 = 1$, In other words, SPT is the only efficient solution then
r = optimal value $- LB = \sum_{i\in N} c_i(SPT) + \sum_{i\in N} Ti(opt.) - \sum_{i\in N} c_i(SPT) - T_{\max}(opt.) = \sum_{i\in N} Ti(opt.) - T_{\max}(opt.) = \sum_{i\in N} Ti(SPT) - T_{\max}(SPT) \ge 0$. Thus $N_1 - 1 \le r \le N_2^* + 1$.
That is $r \in [N_1 - 1, N_2^* + 1]$, Consequently, the theorem holds valid for $N_1 = 1$. ■

If $N_1 = 2$, i.e, With SPT and σ, there are two efficient solutions., say. $N_1 = 2$ implies that $N_1 - 1 = 1$, in case of SPT is optimal then, $r = \sum_{i\in N} c_i(SPT) + \sum_{i\in N} Ti(SPT) - \sum_{i\in N} c_i(SPT) + T_{\max}(opt.) = \sum_{i\in N} Ti(SPT) - T_{\max}(SPT)$. The condition is it must be ≥ 1.

In case of σ is optimal then $r = \sum_{i\in N} c_i(\sigma) + \sum_{i\in N} T_i(\sigma) - \sum_{i\in N} c_i(SPT) - T_{\max}(\sigma) = \sum_{i\in N} c_i(\sigma) - \sum_{i\in N} c_i(SPT) + \sum_{i\in N} T_i(\sigma) - T_{\max}(\sigma)$. Since $\sum_{i\in N} c_i(\sigma) - \sum_{i\in N} c_i(SPT) \ge 1$. So, no condition on $\sum_{i\in N} T_i(\sigma) - T_{\max}(\sigma)$ is necessary. so $r \in [N_1 - 1, N_2^* + 1]$ Thus, theorem holds valid for $N_1 = 2$. ■

If $N_1 = 3$, i.e., Three efficient solutions are available SPT, σ, and $\sigma 1$, say. $N_1 = 3 \to N_1 - 1 = 2$,
in case of SPT is optimal, then $r = \sum_{i\in N} c_i(SPT) + \sum_{i\in N} T_i(opt.) - \sum_{i\in N} c_i(SPT) - T_{\max}(opt.) = \sum_{i\in N} T_i(SPT) - T_{\max}(SPT)$. The condition is it must be ≥ 2. In case of σ is optimal , then $r = \sum_{i\in N} c_i(\sigma) + \sum_{i\in N} T_i(\sigma) - \sum_{i\in N} c_i(SPT) - T_{\max}(\sigma) = \sum_{i\in N} c_i(\sigma) - \sum_{i\in N} c_i(SPT) + \sum_{i\in N} T_i(\sigma) - T_{\max}(\sigma)$. Since $\sum_{i\in N} c_i(\sigma) - \sum_{i\in N} c_i(SPT) \ge 1$. So, the condition is it must be ≥ 1. Finally, in the case of σ_1 is optimal, then $r = \sum_{i\in N} c_i(\sigma_1) + \sum_{i\in N} T_i(\sigma_1) - \sum_{i\in N} c_i(SPT) - T_{\max}(\sigma_1)$. Since $\sum_{i\in N} c_i(\sigma_1) + \sum_{i\in N} c_i(SPT) \ge 2$. So, no condition on $\sum_{i\in N} T_i(\sigma_1) - T_{\max}(\sigma_1)$. Hence $N_1 - 1 \le r \le N_2^* + 1$ or $r \in [N_1 - 1, N_2^* + 1]$. Thus, the theorem is true for $N_1 = 3$. ■

If $N_1 = 4$, i.e., there are four efficient solutions SPT, σ , σ_1, σ_2 say. $N_1 = 4 \to N_1 - 1 = 3$,
In case of SPT is optimal, then $r = \sum_{i\in N} c_i(SPT) + \sum_{i\in N} T_i(SPT) - \sum_{i\in N} c_i(SPT) - T_{\max}(SPT) = \sum_{i\in N} T_i(SPT) - T_{\max}(SPT)$. The condition is it must be ≥ 3. In case of σ is optimal, then $r = \sum_{i\in N} c_i(\sigma) + \sum_{i\in N} Ti(\sigma) - \sum_{i\in N} c_i(SPT) - T_{\max}(\sigma)$. Since $\sum_{i\in N} c_i(\sigma) - \sum_{i\in N} c_i(SPT) \ge 1$. So, the condition is it must be ≥ 2. In case of σ_1 is optimal, then $r = \sum_{i\in N} c_i(\sigma_1) + \sum_{i\in N} T_i(\sigma_1) - \sum_{i\in N} c_i(SPT) - T_{\max}(\sigma_1)$. Since $\sum_{i\in N} c_i(\sigma_1) - \sum_{i\in N} c_i(SPT) \ge 2$. So, $\sum_{i\in N} T_i(\sigma_1) - T_{\max}(\sigma_1)$ must be ≥ 1. Finally, for σ_2 is optimal, then $r = \sum_{i\in N} c_i(\sigma_2) + \sum_{i\in N} T_i(\sigma_2) - \sum_{i\in N} c_i(SPT) - T_{\max}(\sigma_2)$. Since $\sum_{i\in N} c_i(\sigma_2) - \sum_{i\in N} c_i(SPT) \ge 3$. So, no more condition needs. Hence $N_1 - 1 \le r \le N_2^* + 1$ or $r \in [N_1 - 1, N_2^* + 1]$. Thus, the theorem is true for $N_1 = 4$. ■

By the same manner, we continue for k-1 efficient solutions. Table 1 shows the cases for different optimal solutions. We see that if the SPT gives optimal solution, then N_2^* has no condition in the case of one efficient solution, also SPT gives optimal if $N_2^* \ge 1$ in the case of number of efficient solutions is equal to 2. Ans so on for other possibilities.

Table 1. Conditions on N_2^* for different efficient solutions

$\mathbf{N_1}$	SPT	σ	σ_1	σ_2	σ_3	.	.	.	σ_{k-1}
1	-								
2	$N_2^* \geq$	-							
3	$N_2^* \geq$	$N_2^* \geq 1$	-						
4	$N_2^* \geq 3$	$N_2^* \geq 2$	$N_2^* \geq 1$	-					
5	$N_2^* \geq 4$	$N_2^* \geq 3$	$N_2^* \geq 2$	$N_2^* \geq 1$	-				
6	$N_2^* \geq 5$	$N_2^* \geq 4$	$N_2^* \geq 3$	$N_2^* \geq 2$	$N_2^* \geq 1$	-			
.	.	.	.	.	.	.	-		
.	.	.	.	.	.	.	.	-	
$\boldsymbol{k-1}$	$N_2^* \geq k-2$	$N_2^* \geq k-3$	$N_2^* \geq k-4$	$N_2^* \geq k-5$	$N_2^* \geq k-6$	.	.	.	-

Examples

1- Suppose we have the following 4-job problem

n	**1**	**2**	**3**	**4**
p_i	7	9	10	15
d_i	20	15	12	10

At first, by enumeration method the optimal solution is (1, 2, 3, 4), with $\sum_{i\in N} C_i(OPT)$+$\sum_{i\in N} T_i(OPT)$ = 90 + 46 = 136. LB = $\sum_{i\in N} C_i(SPT)$+$T_{max}(opt.)$ = 90 + 31 = 121. N_1= 3 (efficient solutions). Since $\sum_{i\in N} T_i$(SPT)-$T_{max}(SPT) \geq 2$. $N*_2$= $\sum_{i\in N} T_i(SPT)$-$T_{max}(SPT)$ = 46 – 31 = 15, $N*_2$+1 = 16. Therefore r ∈ [2,16].

2- Consider the following 3-job problem

n	**1**	**2**	**3**
p_i	2	10	6
d_i	9	10	20

We apply the theorem in details. 3-jobs means we have 3!= 6 possible schedules. The aim is to find the best schedule according to mentioned objective function. SPT schedule is (1, 3, 2)

n	**1**	**3**	**2**
p_i	2	6	10
d_i	9	20	10
c_i	2	8	18
T_i	0	0	8

LB = $\sum_{i\in N} C_i(SPT)$+$T_{max}(opt.)$ = 28 + 2 = 30. From the 6 schedules 2 of them are efficient.

Schedule 1, (1, 2, 3), $(\sum_{i\in N} C_i(1), T_i(1))$ = (32, 2) efficient solution

Schedule 2, (1, 3, 2), $(\sum_{i\in N} C_i(2), T_i(2))$ = (28, 8) efficient solution

Schedule 3, (2, 1, 3), $(\sum_{i\in N} C_i(3), T_i(3))$ = (40, 3)
Schedule 4, (2, 3, 1), $(\sum_{i\in N} C_i(4), T_i(4))$ = (44, 9)
Schedule 5, (3, 1, 2), $(\sum_{i\in N} C_i(5), T_i(5))$ = (32, 8)
Schedule 6, (3, 2, 1), $(\sum_{i\in N} C_i(6), T_i(6))$ = (40, 9).

$N*_2 = \sum_{i\in N} T_i(SPT)$-$T_{max}(SPT)$ = 8-8 = 0, so $N*_2 + 1$ = 1, Therefore $r \in [1, 1]$. Here, by enumeration method the optimal solution is $\sigma = (1, 2, 3)$, with $\sum_{i\in N} C_i(\sigma)$+$\sum_{i\in N} T_i(\sigma)$ = 32 + 2 = 34. Therefore, r + LB must equal to optimal value 34. So, r = 4 which does not satisfy $r \in [1, 1]$. Because from Table 1, the condition is $N_2^* = \sum_{i\in N} Ti(opt.) - T_{\max}(opt.) \geq 1$, which is here 0.

6. Conclusions and Suggestions

We conclude at the end of this work that one of the key elements in understanding the objective function's nature and the approach taken to solve the problem is the lower bound of the problem. Additionally, efficient solutions are employed to discover the best answer; nevertheless, the relationship between them and our objective function will open up a new field of study: the difference between the optimal value and the lower bound using efficient solutions. In order to answer any problem of this kind, this subject opens algebraic operations and notions. The idea can be used for deriving lower bounds for branch and bound techniques, heuristics method. Finally, using this objective function's new lower bound undoubtedly produces additional outcomes.

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نظرية جديدة للحدود الدنيا في مشاكل الجدولة متعددة الأهداف من نوع NP-Hard

حسن عزيز محمود[1] , اياد محمد رمضان[2]

[1]قسم الرياضيات، كلية التربية، جامعة كرميان، إقليم كردستان، العراق،[2]قسم الرياضيات، كلية العلوم، جامعة السليمانية، إقليم كردستان، العراق

الخلاصة: على جهاز واحد، يجب معالجة كل من (ن) عمل بشكل مستمر . عند الوقت صفر، تكون كل عمل جاهزة للمعالجة. تتمثل المهمة في ايجاد ترتيب يقلل من إجمالي مجموع أوقات الإكمال بالإضافة إلى مجموع التأخير. هذه المسالة غير مقيدة بخوارزمية متعددة الحدود بسبب الثانية. نقدم نظرية توضح العلاقة بين الحل الأمثل والحدود الدنيا وعدد الحلول الفعالة. الحالة هي أن النظرية تعمل على هذه المسائل، بينما كانت الأعمال السابقة تعمل على مسائل مقيدة بخوارزمية متعددة الحدود. هذه النظرية تحدد نطاق الحد الأدنى، وهو أمر بالغ الأهمية لتحديد أفضل إجابة. بالإضافة إلى ذلك، تسمح النظرية باكتشاف حدود دنيا جديدة من خلال فتح مفاهيم جبرية.

الكلمات المفتاحية: حلول ذات حد أدنى، متعددة الأهداف، حلول كفؤة , حل امثل.